

# Critical plane anisotropy adapted for general 3-D stress conditions

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## 1 INTRODUCTION

The shear strength of soils is almost invariably transversely anisotropic due to bedding planes and layering. The critical plane approach to shear strength anisotropy has been shown to provide a simple and realistic description of transversely anisotropic soils. However, the technique, based on the orientation of the plane of maximum stress obliquity, is essentially two-dimensional, neglecting the influence of the intermediate principal stress. When the magnitude of the intermediate principal stress approaches the major or minor principal stress (triaxial extension or triaxial compression) critical plane anisotropy can produce erratic results due to abrupt shifts in the orientation of the plane of maximum obliquity. For this reason, the use of the technique has been mostly restricted to two dimensions.

In this paper, an attempt is made to adapt critical plane anisotropy for general, three dimensional stress states, using a simple averaging procedure for stress states approaching triaxial extension or compression. An elastic, perfectly plastic constitutive model is developed to simulate the drained shear strength of anisotropic soils. Performance of the model is evaluated through comparisons with laboratory test data for anisotropic soils.

## 2 CRITICAL PLANE ANISOTROPY

The critical plane approach to modeling anisotropy (Miura et al. 1986, Pietruszczak & Mroz 2001) employs an isotropic yield surface, but with the size of the yield surface adjusted as a function of the orientation of the plane of maximum stress obliquity (the critical plane). Shear strength is a function of the angle  $\delta$  between the critical plane and a pre-defined weak plane, with minimum strength occurring when the critical plane coincides with the weak plane. For a given stress state, the orientation of the critical plane can be determined from the directions of the principal stresses: The critical plane is perpendicular to the  $\sigma_1$ - $\sigma_3$  plane, and is at an angle of

$$\theta = \frac{\pi}{4} - \frac{\phi_{ref}}{2} \quad (1)$$

from the  $\sigma_1$  direction. In this relation,  $\phi_{ref}$  is a fixed, reference friction angle. These relations are illustrated in Figure 1. Note that  $\sigma_1$  is the most compressive principal stress, while  $\sigma_3$  the least compressive stress.

Various descriptions of the variation in strength with orientation are possible. For instance, Miura et al. (1986) obtained an excellent match to their laboratory data from hollow cylinder torsional shear tests on dense anisotropic sand using the relation:

$$\sin \phi_{isotropic} = \sin \phi_0 - \alpha \cos 2\delta \quad (2)$$

where  $\phi_{isotropic}$  is the friction angle to be used in the isotropic yield condition, while  $\alpha$  and  $\phi_0$  are material properties. Equation 2 is roughly equivalent to the more formal relation of Pietruszczak & Mroz (2001),

who describe the variation in strength with orientation of the critical plane using a 2<sup>nd</sup> order symmetric tensor.

Since the technique relies on the orientation of the  $\sigma_1$  and  $\sigma_3$  directions, it is best suited for stress states where these directions are unambiguous, in other words, where the intermediate principal stress  $\sigma_2$  is not too close to either  $\sigma_1$  or  $\sigma_3$ . The configuration of the intermediate principal stress is often described in terms of the parameter:

$$b = \frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_3)} \quad (3)$$

For stress states near  $b = 0$  (triaxial compression)  $\sigma_2$  is almost equal to  $\sigma_3$ , and a small shift in stresses can switch the order of the principal stresses so that the  $\sigma_3$  direction abruptly switches to the previous  $\sigma_2$  direction. This changes the orientation of the critical plane which can produce an abrupt change in shear strength. A similar problem occurs at stress states near  $b = 1$  (triaxial extension), where the  $\sigma_1$  direction can abruptly switch to the previous  $\sigma_2$  direction. Thus, critical plane anisotropy performs erratically for stress states near  $b = 0$  and  $b = 1$ .

For this reason, use of the technique has been mostly restricted to two dimensions, e.g. plane strain where  $b$  is generally in the range of 0.2 to 0.35 or laboratory tests at a specified  $b$ . For instance, the hollow cylinder torsional shear tests of Miura et al. (1986) were performed at  $b = 0.5$  ( $\sigma_2$  at the midpoint between  $\sigma_1$  and  $\sigma_3$ ).

Nonetheless, given the simplicity and accuracy of critical plane anisotropy, along with the clear physical meaning of the strength parameters, it appears worthwhile to attempt to adapt the technique for more general stress conditions. The alternative is to resort to the more abstract, formal approach based on joint invariants of the stress tensor and a fabric tensor. Pietruszczak & Mroz (2001) review these techniques and show that critical plane anisotropy can be roughly approximated using a fourth order fabric tensor (but not with the commonly used 2<sup>nd</sup> order fabric tensor). They conclude that the critical plane approach is more convenient and gives results more consistent with experimental data.

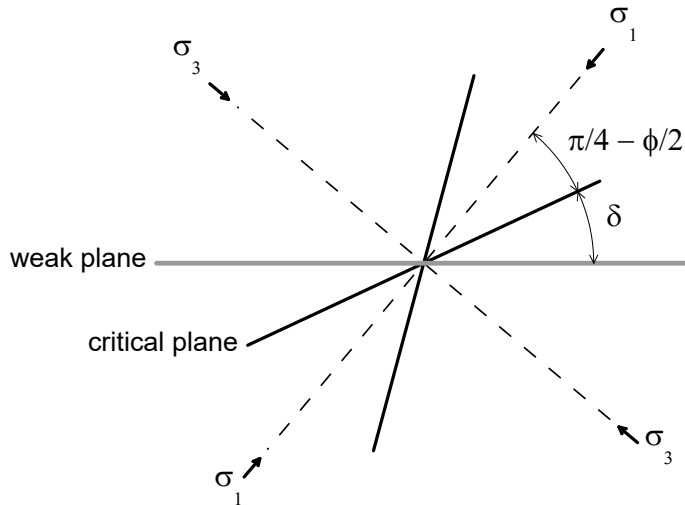


Figure 1. Orientation of the critical plane in relation to the  $\sigma_1$  and  $\sigma_3$  directions, and to the weak plane.

### 3 CRITICAL PLANE ANISOTROPY ADAPTED FOR GENERAL STRESS CONDITIONS

To address the limitations of critical plane anisotropy described above, an attempt is made here to adapt the technique for general three-dimensional stress states using a simple averaging procedure for stress states with  $b$  near 0 or 1. To facilitate detailed comparisons with laboratory tests data, the model is developed using the Lade yield condition (Lade & Duncan 1975)

$$\frac{I_1^3}{I_3} - \kappa = 0 \quad (4)$$

where  $I_1$  and  $I_3$  are the first and third invariants of the stress tensor and  $\kappa$  is a strength parameter. The Lade surface is coincident with the Mohr-Coulomb surface at the triaxial compression point if

$$\kappa = \frac{(N_\phi + 2)^3}{N_\phi} \quad (5)$$

where

$$N_\phi = \frac{1 + \sin \phi}{1 - \sin \phi} \quad (6)$$

and  $\phi$  is the friction angle for triaxial compression. In what follows, the friction angle for triaxial compression is used as the strength parameter for the Lade surface, rather than the less familiar  $\kappa$  parameter.

The anisotropic strength of the model is described in terms of the maximum and minimum friction angles,  $\phi_{max}$  and  $\phi_{min}$ , along with the orientation of the weak plane. For computing the orientation of the critical plane,  $\phi_{min}$  is used as the reference friction angle  $\phi_{ref}$  in Equation 1. The variation in strength as a function of the angle  $\delta$  between the critical plane and the weak plane is specified with Equation 2, using the constants:

$$\alpha = \frac{\sin \phi_{max} - \sin \phi_{min}}{1 + \sin \phi_{min}} \quad (7)$$

and

$$\sin \phi_0 = \sin \phi_{min} + \alpha \quad (8)$$

To adapt the critical plane technique for general stress states we first consider stress states with  $b \leq 0.5$ . For these conditions a shear strength  $\phi_{13}$  is first computed in the usual way (Eq. 2) from the orientation of the critical plane constructed from the  $\sigma_1$  and  $\sigma_3$  directions. A second shear strength  $\phi_{23}$  is then computed from the orientation of a critical plane constructed from the  $\sigma_2$  and  $\sigma_3$  directions. Finally, a composite strength is computed as a weighted average of these two strengths.

$$\phi_{isotropic} = w_1 \phi_{13} + w_2 \phi_{23}, \quad b \leq 0.5 \quad (9)$$

where  $w_1$  and  $w_2$  are weighting factors such that  $w_1 + w_2 = 1.0$ . For stress states with  $b > 0.5$ , a second shear strength  $\phi_{12}$  is computed with from the orientation of the critical plane constructed from the  $\sigma_1$  and  $\sigma_2$  directions, and the composite strength is given by

$$\phi_{isotropic} = w_1 \phi_{13} + w_2 \phi_{12}, \quad b > 0.5 \quad (10)$$

The weighting factors are a function of the parameter  $b$  such that

$$w_1 = \frac{1}{2} \sin(b\pi) + \frac{1}{2} \quad (11)$$

For  $b = 0.5$ ,  $w_1 = 1$  so that  $\phi_{isotropic} = \phi_{13}$ , recovering the original critical plane formulation. However, at  $b = 0$  and  $b = 1$ ,  $w_1 = 0.5$ , and the shear strength is an equally weighted average of  $\phi_{13}$  and  $\phi_{23}$  or of  $\phi_{13}$  and  $\phi_{12}$ . Clearly the selection of the interpolation function, Equation 11, is somewhat arbitrary, as is the assumption of equal weighting at  $b = 0$  and  $b = 1$ . It could be argued that the lower of the two shear strengths should be given a higher weight.

#### 4 COMPARISONS WITH LABORATORY TESTING DATA

Response of the critical plane model is first compared to data from the hollow cylinder torsional shear tests on dense Toyoura Sand performed by Miura et al. (1986). The tests specimens are anisotropic due to the particle fabric developed during the sample preparation process under gravity. During these tests the intermediate stress parameter  $b$  is held constant at 0.5 ( $b = 0.5$ ), so that the strength averaging modifications described above do not come into play. Instead, this comparison demonstrates the response of the critical plane model under the ideal conditions for which it was originally developed.

In these hollow cylinder tests, the inclination of the major principal stress  $\sigma_1$  with respect to the axial ( $z$ ) direction was controlled by the ratio of axial and torsional shear stresses applied at the ends of the cylinder. Tests were run for  $\sigma_1$  inclinations ranging from 0 to 90 degrees. Figure 2 shows the Miura et al. (1986) test data presented in terms of  $\sigma_{\theta\theta}$ ,  $\sigma_{rr}$  and  $\sigma_{zz}$ , corresponding to circumferential, radial and axial stresses respectively. Note that in the stress space employed, the angle,  $\beta$ , between  $\sigma_1$  and the vertical axis is twice the physical inclination of  $\sigma_1$ . At Point A,  $\sigma_1$  is vertical, while at Point B,  $\sigma_1$  is inclined at  $45^\circ$  and at Point C,  $\sigma_1$  is horizontal.

Also shown in Figure 2 is a critical plane anisotropy fit using  $\phi_{max} = 54^\circ$  and  $\phi_{min} = 39^\circ$ , with the orientation of the weak plane set to horizontal. For reference, the isotropic Lade surfaces for  $\phi = 54^\circ$  and  $\phi = 39^\circ$  are also plotted. The strength of the anisotropic model is bracketed by these two surfaces. For the selected properties, critical plane theory predicts that the minimum shear strength should occur when  $\sigma_1$  is inclined at an angle of  $\beta = 45 + \phi_{ref}/2 = 64.5^\circ$  or  $2\beta = 129^\circ$  in the plotted stress space. This can be seen to agree well with the test data.

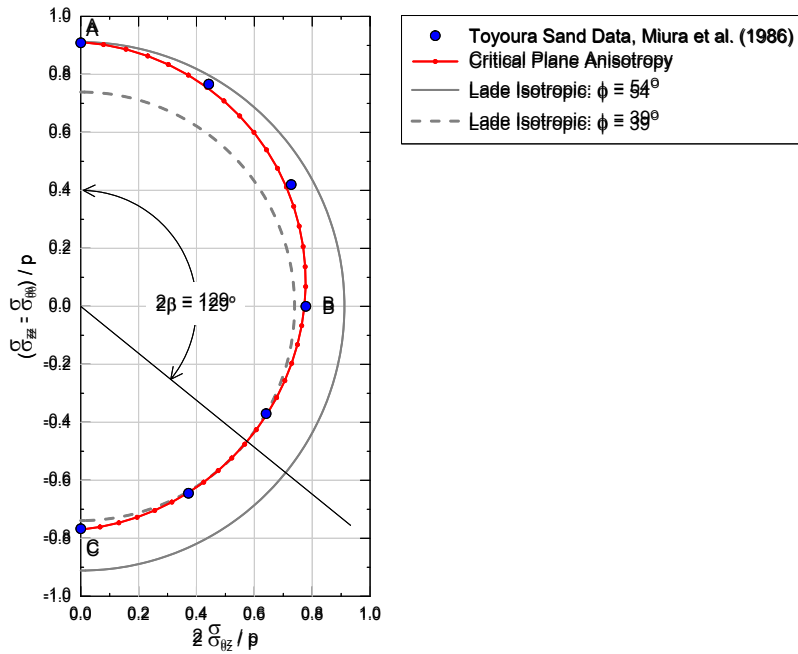


Figure 2. Critical plane anisotropy fit for the hollow cylinder torsional shear data of Miura et al. (1986).

Performance of the critical plane model is next compared to the true triaxial test data for Hostun RF Sand of Jafarzadeh et al. (2008), presented in Figure 3. These tests consisted of radial stress probes in the deviatoric plane with  $p = 200$  kPa. Performed at various  $b$  values, the stress probe data make possible an evaluation of the procedure described in Section 3 for adapting critical plane anisotropy for stress states with  $b$  approaching 0 or 1. As shown in the figure, a model fit using  $\phi_{max} = 35.5^\circ$  and  $\phi_{min} = 27^\circ$ , with the orientation of the weak plane set to horizontal provides a reasonable match to the lab data. For reference, the isotropic Lade surface for  $\phi = 35.5^\circ$  is also shown.

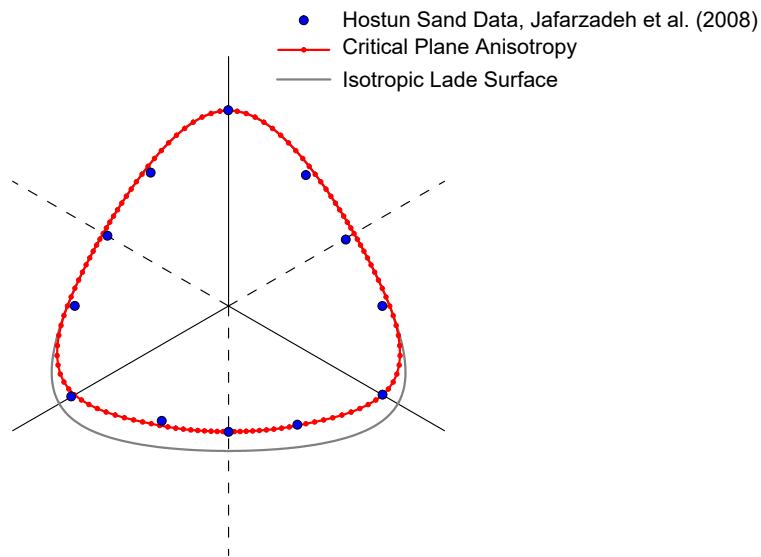


Figure 3. Critical plane anisotropy fit to the true triaxial test data of Jafarzadeh et al. (2008).

## 5 CONCLUSION

Critical plane anisotropy provides a simple and realistic description of transversely anisotropic soils. However, practical use of this technique has been limited due to its essentially 2-D nature. The scheme presented here for adapting critical plane anisotropy to more general stress conditions gives results in reasonable agreement with laboratory tests on transversely isotropic soils. Although the scheme is somewhat ad hoc, it preserves as much as possible the simplicity and mechanical basis of the critical plane approach.

The degree of anisotropy shown in these experiments is relatively modest, as would be expected for carefully constructed, homogenous laboratory specimens, where the anisotropy is an almost inadvertent side effect of the sample preparation process under gravity. Anisotropy of natural soils is typically much more significant due to depositional layering at multiple length scales, a feature also seen in mine tailings and hydraulic fill. Other man-made soils, such as compacted fills, similarly show significant strength anisotropy.

## REFERENCES

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